Differential Geometry I: Curves and surfaces

3.1 Express the length of the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
in the form of an integral (do not attempt to calculate this integral, as it cannot be calculated by elementary means; this integral gave rise to the theory of elliptic functions).

Solution. Parametrize the ellipse by $(x(t), y(t)) = (a\cos t, b\sin t)$. Assuming $0 < b \leqslant a$, its length is
$$L=\int_0^{2\pi}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,dt = a\int_0^{2\pi}\sqrt{1-k^2\cos^2 t}\,dt,$$
with $k=\sqrt{1-\frac{b^2}{a^2}}$. This is an elliptic integral of the second kind.

Alternatively, parametrizing the half-ellipse as a graph $y = f(x) = b\sqrt{1-x^2/a^2}$, one gets
$$\frac{1}{2}L=\int_{-a}^{a}\sqrt{1+f'(x)^2}\,dx=\int_{-a}^{a}\sqrt{\frac{a^2-k^2x^2}{a^2-x^2}}\,dx \overset{z=x/a}{=} a\int_{-1}^{+1}\sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz.$$

3.2 The *astroid* is the plane curve with equation
$$|x|^{\frac{2}{3}}+|y|^{\frac{2}{3}}=1.$$

(a) Draw the astroid.

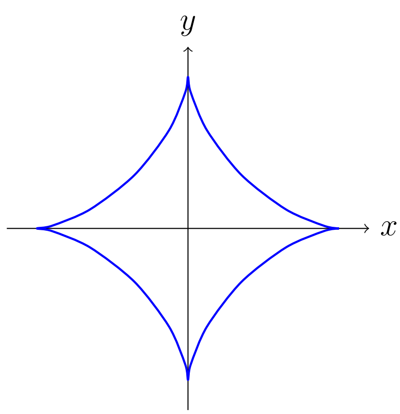
(b) Find a parametrization of the astroid. What are the singular points?

(c) Compute the length of one cycle of the astroid.

(d) Compute the arc length parameter for the part of the curve in the first quadrant with initial point at $(1, 0)$.

(e) Find the natural parametrization for the part of the curve in the first quadrant with the same initial point.

Solution. (a) In the quadrant $x, y \geqslant 0$ the arc is $y = (1-x^{2/3})^{3/2}$. The rest follows by symmetry.



(b) Setting $\xi = \sqrt[3]{x}, \eta = \sqrt[3]{y}$ gives $\xi^2 + \eta^2 = 1$, parametrized by $(\cos u, \sin u)$. So

$$\alpha(u) = (\cos^3 u, \sin^3 u), \quad 0 \leqslant u \leqslant 2\pi,$$

is a parametriation of the astroid. We calculate its speed:

$$V_\alpha(u) = 3|\cos u \sin u| = \tfrac{3}{2}|\sin 2u|.$$

Since the functions defining the parametrization are smooth functions, the singular points are precisely where the speed vanishes, namely at $u = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (which correspond to the points $(1,0), (0,1), (-1,0)$ and $(0,-1)$.

Remark. These points correspond to the "cusps" of the astroid, and would be singular points for any parametriation.

(c) Given the formula for the speed computed before,we have:

$$L = \int_0^{2\pi} \frac{3}{2}|\sin 2u|\, du = \frac{3}{4}\int_0^{4\pi} |\sin t|\, dt = 6.$$

(d) For $u \in [0, \pi/2]$, we have:

$$s(u) = \tfrac{3}{2}\int_0^u \sin(2\tau)\, d\tau = \tfrac{3}{2}\sin^2 u.$$

Though the exercise doesn't ask for it, if we wanted to compute the arc length parameter at every quadrant, we could argue as follows: Due to the rotational symmetry of the astroid, the arc length parameter for $u \in [k\frac{\pi}{2}, (k+1)\frac{\pi}{2}]$, $k = 0,1,2,3$, with initial point at $u = k\frac{\pi}{2}$, has the same expression as above, but with $u - k\frac{\pi}{2}$ in place of u. Therefore, since the length of each arch of the astroid is equal to $1/4$ the total length, we have that, for $k = 0,1,2,3$, if $u \in [k\frac{\pi}{2}, (k+1)\frac{\pi}{2}]$:

$$s(u) = \tfrac{3}{2}\sin^2(u - k\frac{\pi}{2}) + k\frac{L}{4} = \tfrac{3}{2}\sin^2(u - k\frac{\pi}{2}) + \frac{3}{2}k.$$

(e) Inverting the previous expression, we have for $s \in [0, \frac{3}{2}]$ (corresponding to the parameter of the astroid in the first quadrant of the plane):

$$u(s) = \arcsin\left(\sqrt{\frac{2}{3}s}\right)$$

Therefore, recalling that $\sin(\arcsin x) = x$ and $\cos(\arcsin x) = \sqrt{1-x^2}$, the natural parametrization of the first arch of the astroid is given by:

$$\tilde{\alpha}(s) = \big(\cos^3(u(s)), \sin^3(u(s))\big) = \big((1 - \tfrac{2}{3}s)^{3/2}, (\tfrac{2}{3}s)^{3/2}\big).$$

One can then see that the natural parametrization of the whole astroid can be given as follows (inverting, in each case, the expression for $s(u)$ in each of the four quadrants computed in the previous part, this is a slightly arduous task):

$$\arcsin\left(\sqrt{\frac{2}{3}(s-k)} + \frac{k\pi}{2}\right)$$

$$\tilde{a}(s) \begin{cases} \left((1-\frac{2}{3}s)^{3/2}, (\frac{2}{3}s)^{3/2}\right), \quad s \in [0, \frac{3}{2}], \\ \left(-(\frac{2}{3}s-1)^{3/2}, (2-\frac{2}{3}s)^{3/2}\right), s \in [\frac{3}{2}, 3], \\ \left(-(3-\frac{2}{3}s)^{3/2}, -(\frac{2}{3}s-2)^{3/2}\right), s \in [3, \frac{9}{2}], \\ \left((\frac{2}{3}s-3))^{3/2}, -(4-\frac{2}{3}s)^{3/2}\right), s \in [\frac{9}{2}, 6]. \end{cases}$$

Remark. Note that the above parametrization is continuous but not differentiable at the cusp points $s = 0, \frac{3}{2}, 3, \frac{9}{2}$. In particular, those points are singular in any parametrization (this is always truw for points which are singular in the natural parametrization).

3.3 Let $\gamma : I \to \mathbb{R}^n$ be a twice differentiable regular curve with constant speed. Show that at every point of γ, the acceleration is perpendicular to the velocity.

Solution. Since the speed of γ is constant, we have

$$\|\dot{\gamma}(t)\|^2 = \text{const} \quad \text{for}\, t \in I. \tag{1}$$

Recall that, with respect to the Cartesian coordinates on $\mathbb{R}^n$, we have

$$\|\dot{\gamma}(t)\|^2 = \sum_{i=1}^{n} \left(\dot{\gamma}_i(t)\right)^2,$$

therefore, differentiating (1), we have:

$$0 = \frac{d}{dt}\left(\|\dot{\gamma}(t)\|^2\right) \Rightarrow 0 = \frac{d}{dt}\left(\sum_{i=1}^{n} \left(\dot{\gamma}_i(t)\right)^2\right) = \sum_{i=1}^{n} 2\ddot{\gamma}_i(t)\dot{\gamma}_i(t) = 2\langle \ddot{\gamma}, \dot{\gamma}\rangle.$$

Hence, $\ddot{\gamma} \perp \dot{\gamma}$.

3.4 **(a)** Let (x, y) be Cartesian coordinates on $\mathbb{R}^2$. Recall the precise definition of polar coordinates (r, θ) and specify their domain of definition.

(b) Write the general equation of a line in polar coordinates, then the equation of a circle of radius a and center $c = (r_0, \theta_0)$.

(c) Let $\gamma(t) = (r(t), \theta(t))$ be a C^1 curve written in polar coordinates. Find and prove a formula giving its length in these coordinates.

(d) The logarithmic spiral is the plane curve with polar equation $r = e^{\theta}$. Use the previous formula to compute the length of one cycle of this spiral defined by $0 \leqslant \theta \leqslant 2\pi$. Then find the natural parametrization with initial point $(1, 0)$.

Solution. (a) The polar coordinates (r, θ) are defined on $\mathbb{R}^2 \setminus 0$, so that $r(p)$ is the distance of p from the origin and θ is the angle between the x axis and the vector Op. Note that we will choose θ to take values in $[0, , 2\pi)$, i.e. to have a discontinuity along the positive x semi-axis; we could have chosen any other semi-infinite line from the origin for convenience (corresponding to a discontinuity at a certain $\theta = \theta_1$). In particular, we have

$$r = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} \arccos(x/\sqrt{x^2+y^2}), & y \geqslant 0, \\ 2\pi - \arccos(x/\sqrt{x^2+y^2}), & y < 0. \end{cases}$$

Note that the functions $r(x, y), \theta(x, y)$ are smooth on $\mathbb{R}^2 \setminus \{x \geqslant 0, y = 0\}$, so this is where they define a smooth coordinate transformation. Inverse transformation:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

(b) Line: $ax + by + c = 0$ in cartesian coordinates (with not both a, b equal to 0), therefore (using the formula for (x, y) in terms of (r, θ) above):

$$ar\cos\theta + br\sin\theta + c = 0.$$

Circle: $(x - x_0)^2 + (y - y_0)^2 = a^2 \iff x^2 + y^2 - 2x_0 x - 2y_0 y + x_0^2 + y_0^2 - a^2 = 0$ in Cartesian coordinates, therefore:

$$r^2 - 2r_0 r\cos\theta_0\cos\theta - 2r_0 r\sin\theta_0\sin\theta + r_0^2 - a^2 = 0,$$

which can be reexpressed as

$$r^2 - 2r_0 r\cos(\theta - \theta_0) + r_0^2 - a^2 = 0.$$

(c) For $\alpha(t) = (r(t), \theta(t))$ in polar coordinates, we calculate its expression in Cartesian coordinates:

$$\tilde{\alpha}(t) = (x(t), y(t)) = (r(t)\cos(\theta(t)), r(t)\sin(\theta(t))).$$

Therefore, the velocity in Cartesian coordinates is

$$\frac{d\tilde{\alpha}}{dt}(t) = \Big(\dot{r}(t)\cos(\theta(t)) - r(t)\dot{\theta}(t)\sin(\theta(t)),\ \dot{r}(t)\sin(\theta(t)) + r(t)\dot{\theta}(t)\cos(\theta(t))\Big)$$

and, thus, the speed is

$$V(t) = \left\|\frac{d\tilde{\alpha}}{dt}(t)\right\| = \sqrt{\dot{r}^2(t) + \Big(r(t)\dot{\theta}(t)\Big)^2}.$$

Thus

$$\ell = \int_{t_0}^{t_1} \sqrt{\dot{r}^2(t) + \Big(r(t)\dot{\theta}(t)\Big)^2}\, dt.$$

(d) If we parametriz the spiral by the angle θ, then it takes the form

$$\beta : \theta \to (r, \theta) = (e^{\theta}, \theta).$$

Therefore, its speed is

$$V_\beta(\theta) = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + \left(r\frac{d\theta}{d\theta}\right)^2} = \sqrt{2}e^\theta.$$

Therefore, the arc length parameter with initial point at $\theta = 0$, which is the point $(1, 0)$ in polar (but also in Cartesian) coordinates, is

$$s(\theta) = \int_0^\theta V_\beta(\theta)\, d\theta = \sqrt{2}(e^\theta - 1).$$

The length of the cycle between $0 \leqslant \theta \leqslant 2\pi$ is

$$L = s(2\pi) = \sqrt{2}(e^{2\pi} - 1).$$

The natural length parametrization is obtained by inverting $s(\theta)$:

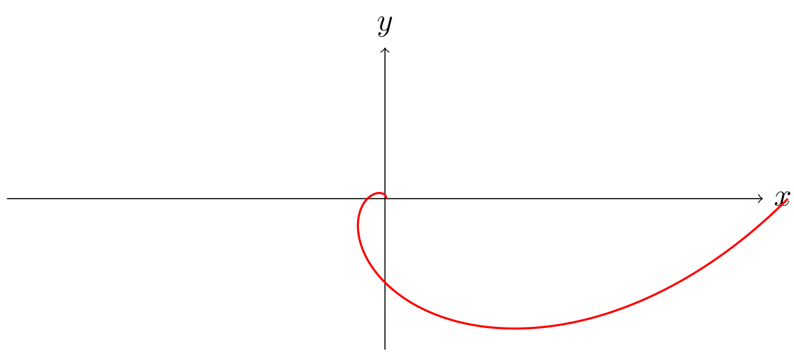
$$\theta(s) = \log\left(\frac{s}{\sqrt{2}} + 1\right).$$

In terms of s, the curve therefore takes the following form in polar coordinates:

$$\tilde{\beta}(s) = \beta(\theta(s)) = (r(\theta(s)), \theta(s)) = \left(e^{\theta(s)}, \theta(s)\right) = \left(\frac{s}{\sqrt{2}} + 1, \log\left(\frac{s}{\sqrt{2}} + 1\right)\right).$$

In Cartesian coordinates:

$$(x(s), y(s)) = (1 + s/\sqrt{2})\big(\cos(\log(1 + s/\sqrt{2})), \sin(\log(1 + s/\sqrt{2}))\big).$$



3.5 The *conchoid of Nicomedes* (belonging to the broader family of conchoid curves) is the curve $\mathcal{C}$ in the Euclidean plane defined as follows: Fix a $b \geqslant 0$, a point O in the plane (called the *pole*) and a line D (called the *directrix*) not passing through O. For any point p in the plane such that $p \notin D$ and $p \neq O$, let

$$f(p) = d(p, q),$$

where q is the intersection of D with the line through O and p, i.e.

$$q = (O + \mathbb{R}\,\overrightarrow{Op}) \cap D.$$

Then the curve $\mathcal{C}$ is defined as the level set

$$\mathcal{C} = \{p \in \mathbb{R}^2 \mid f(p) = b\}.$$

(a) Draw the curve $\mathcal{C}$. Is it connected?

(b) Give a polar equation of this curve (assume that the directrix is vertical and that the pole is the origin).

(*c) Nicomedes (3^{rd} century BC) used this curve to solve the problem of trisection of an angle (which is of course not possible just with a ruler and compass). This was his process: Suppose that $\widehat{AOB} = \theta_0$ is the angle we want to trisect (and assume that $\theta_0 < \frac{\pi}{2}$).

* Draw a line m perpendiculat to OA. Let P be the point of interection of m with OB.
* Let $\mathcal{C}$ be the conchoid curve with pole O, directrix m and $b = 2d(O,P)$. Keep the connected component of the curve that lies in the half plane defined by m not containing O.
* Draw a line from P which is parallel to OA. Let M be the intersection point with $\mathcal{C}$.

Then $\widehat{AOM} = \frac{1}{3}\theta_0$. Prove this statement.

Solution. (a) The curve $\mathcal{C}$ is not connected (when $b > 0$), since it consists of two branches: For each q on the line D, there are exactly two points along the line Oq at distance b from q.

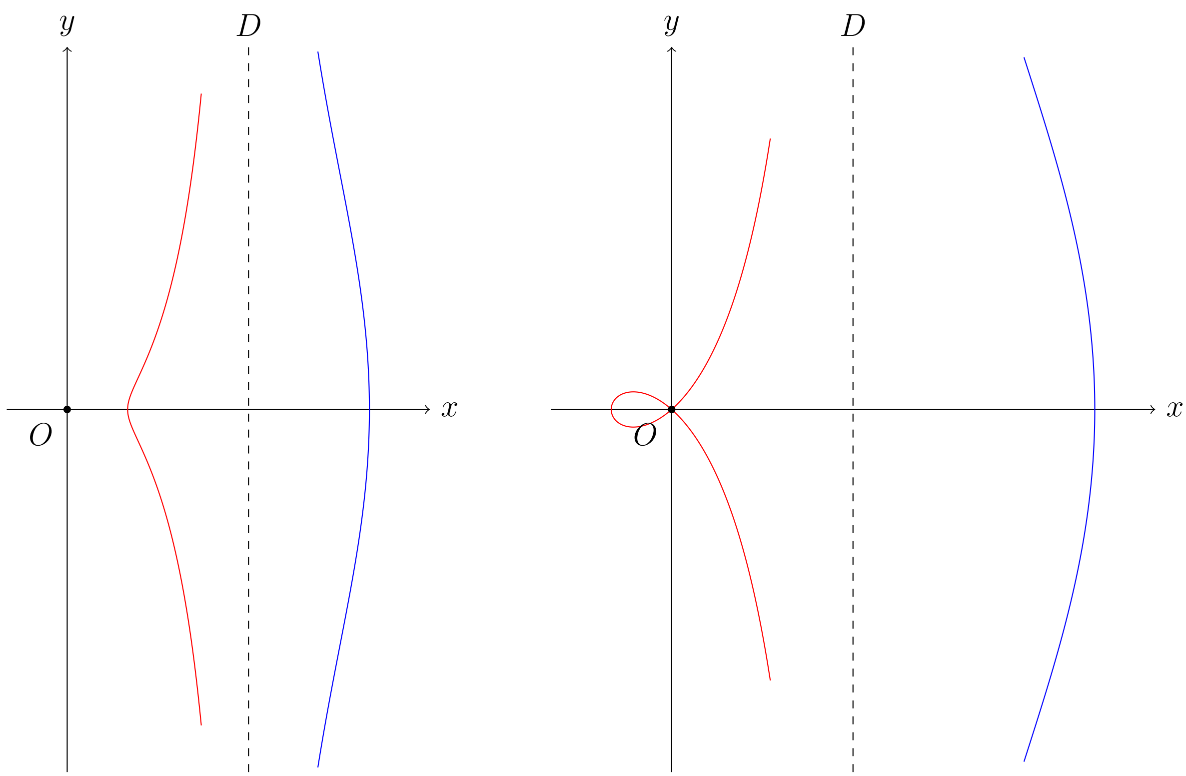


Figure 1: The figure on the left depicts a conchoid with $b < d(O,D)$; the one on the right corresponds to $b > d(O,D)$.

(b) When D is vertical, it is of the form $x = a$ (we will assume that $a > 0$. In polar coordinates, this becomes

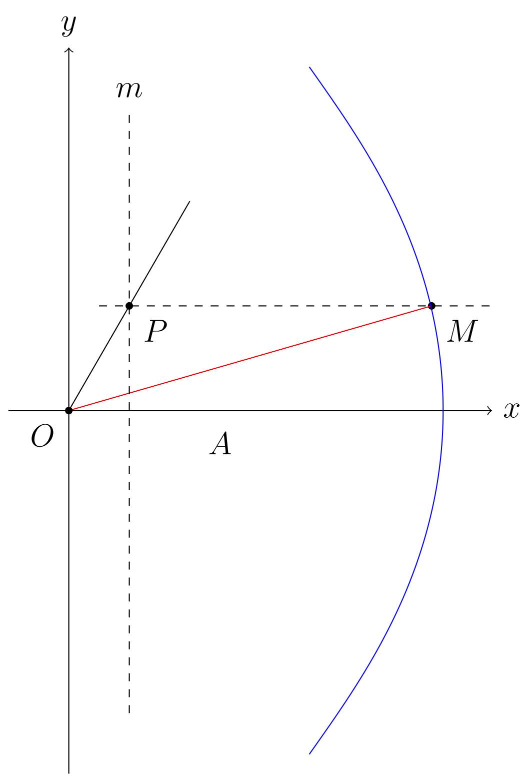
$$r = \frac{a}{\cos(\theta)}.$$

Therefore, the two branches of the conchoid (which is defined by moving in the radial direction from the point on D by distance b) are defined by

$$r = \frac{a}{\cos(\theta)} \pm b.$$

(c) Assuming that O is the origin and OA is along the x axis, the line m is of the form $x = a$ or, in polar coordinates, $r = \frac{a}{\cos(\theta)}$. The point P has to be in polar coordinates of the form (r_0, θ_0) (since the angle of OP with OA is θ_0), therefore, since it lies on the line m, $P = \left(\frac{a}{\cos(\theta_0)}, \theta_0\right)$. In particular,

$$d(O, P) = \frac{a}{\cos(\theta_0)}.$$



Therefore, the branch of conchoid with directrix m and $b = 2d(O, P)$ lying to the side of m not cotaining O is given by the relation

$$\mathcal{C} = \left\{ r = \frac{a}{\cos(\theta)} + \frac{2a}{\cos(\theta_0)} \right\}.$$

On the other hand, the line passing through P which is parallel to OA is of the form $y = \text{const} = y(P) = r_0 \sin(\theta_0) = a\tan(\theta_0)$. In polar coordinates, this line is expressed as

$$r = \frac{a\tan(\theta_0)}{\sin(\theta)}.$$

The intersection of this line with $\mathcal{C}$ is the point $M = (r_1, \theta_1)$; in particular r_1, θ_1, will satisfy both the above relations, namely

$$r_1 = \frac{a}{\cos(\theta_1)} + \frac{2a}{\cos(\theta_0)}$$
$$r_1 = \frac{a\tan(\theta_0)}{\sin(\theta_1)}.$$

Solving for r_1, we get

$$\frac{a}{\cos(\theta_1)} + \frac{2a}{\cos(\theta_0)} = \frac{a\tan(\theta_0)}{\sin(\theta_1)}$$

which, after rearranging (and removing a from both sides) yields

$$\sin(\theta_1)\cos(\theta_0)+2\cos(\theta_1)\sin(\theta_1) = \sin(\theta_0)\cos(\theta_1) \Leftrightarrow \sin(\theta_0)\cos(\theta_1)-\cos(\theta_0)\sin(\theta_1) = 2\cos(\theta_1)\sin(\theta_1) \Leftrightarrow \sin(\theta_0-$$

which, since both θ_0 and θ_1 are less than $\frac{\pi}{2}$, yields

$$\theta_0 - \theta_1 = 2\theta_1 \Leftrightarrow \theta_1 = \frac{1}{3}\theta_0.$$

3.6 Let $F : I \to SO(n) \subset \mathcal{M}_n(\mathbb{R}) = \mathbb{R}^{n\times n}$ be a C^1 curve taking values in the orthogonal group. Prove that $F(t)^{-1}\dot{F}(t)$ and $\dot{F}(t)F(t)^{-1}$ are skew-symmetric matrices for all $t \in I$. *(Recall that a matrix $A \in \mathcal{M}_n(\mathbb{R})$ is skew symmetric if $A^T = -A$.)*

Solution. Differentiate $F(t)F^\top(t) = \mathbb{I}$: Using the fact that the derivative of the product of matrices satisfies the Leibniz rule, we get

$$\dot{F}(t)F^\top(t) + F(t)\dot{F}^\top(t) = 0..$$

IN view of the fact that $(AB)^\top = B^\top A^\top$, we obtain:

$$\dot{F}F^\top + (\dot{F}F^\top)^\top = 0,$$

namely that $\dot{F}F^\top$ is antisymmetric. Since $F^\top = F^{-1}$ (due to our assumption that $FF^\top = \mathbb{I}$, we onfer that $\dot{F}F^{-1}$ is antisymmetric. Similarly, we obtain that $F^{-1}\dot{F}$ is also antisymmetric by noting that $F^\top$ is also orthogonal and repeating the previous process for $F^\top$ in place of F.

3.7 Recall that the exponential $\exp(A)$ of a square matrix $A \in M_n(\mathbb{R})$ is defined by the series

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \cdots$$

This series is known to converge. It is also known that if $AB = BA$, then $\exp(A + B) = \exp(A)\exp(B)$ (the proof is the same as for the exponential of a sum of real numbers).

(a) Show that if $A \in M_n(\mathbb{R})$ is antisymmetric, then $\exp(A) \in SO(n)$.

(b) Compute the matrix $\exp(tJ)$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution. (a) We can readily calculate:

$$(\exp(A))^\top = \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k \right)^\top = \sum_{k=0}^{\infty} \frac{1}{k!} (A^\top)^k = \exp(A^\top),$$

therefore

$$\exp(A)\,(\exp(A))^\top = \exp(A)\exp(A^\top).$$

If A is antisymmetric, then $A^\top = -A$ and hence the above becomes

$$\exp(A)\,(\exp(A))^\top = \exp(A)\exp(-A) = \exp(A - A) = \mathbb{I}$$

(since A trivially commutes with $-A$, the above formula for the product of the exponentials is valid). Therefore, $\exp(A) \in O(n)$. Moreover, one can check that, for any matrix which is diagonal (and any antisymmetric matrix is diagonalizable over $\mathbb{C}$), we have

$$\exp\left(\text{diag}[\lambda_1, \lambda_2, \ldots]\right) = \text{diag}\left[\sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k, \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_2^k, \ldots \right] = \text{diag}[e^{\lambda_1}, e^{\lambda_2}, \ldots]$$

so

$$\det(\exp(A)) = \exp(\text{tr}(A)).$$

Since A is antisymmetric, $\text{tr}(A) = 0$, therefore $\det(\exp(A)) = +1$, i.e. $A \in SO(n)$.

(b) $J^2 = -\mathbb{I}$, $J^3 = -J$, $J^4 = \mathbb{I}$. Hence,

$$J^k = \begin{cases} \mathbb{I}, & k \equiv 0 \mod 4, \\ J, & k \equiv 1 \mod 4, \\ -\mathbb{I}, & k \equiv 2 \mod 4, \\ -J, & k \equiv 3 \mod 4. \end{cases}$$

As a result:

$$\begin{aligned}\exp(tJ) &= \sum_{k=0}^{\infty} \frac{1}{k!} t^k J^k \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{(4k)!} t^{4k} \mathbb{I} + \frac{1}{(4k+1)!} t^{4k+1} J - \frac{1}{(4k+2)!} t^{4k+2} \mathbb{I} - \frac{1}{(4k+3)!} t^{4k+3} J \right) \\ &= \cos t \, I + \sin t \, J,\end{aligned}$$

i.e.

$$\exp(tJ) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

the rotation matrix.

3.8 Prove the following statement or find a counterexample: If $\gamma_n : [a, b] \to \mathbb{R}^n$ is a sequence of curves converging *uniformly* to the curve $\gamma : [a, b] \to \mathbb{R}^n$ (all curves are assumed to be of class C^1), then the lengths converge, i.e.

$$\ell(\gamma) = \lim_{n \to \infty} \ell(\gamma_n).$$

Solution. The statement is false; in general, the length involves $\dot{\gamma}$, so one should also have convergence of the derivatives of γ_n (in some suitable topology) to infer the convergence of the lengths. A counterexample is as follows (note that any counterexample should involve some increasing number of "wiggles" as we approach the limiting curve): Let

$$\gamma_n(x) = (x, \tfrac{1}{\sqrt{n}} \sin(nx)), \quad 0 \leqslant x \leqslant 2\pi$$

and $\gamma(x) = (x, 0)$. Then $\gamma_n(x) \to (x, 0)$ uniformly in x. However, we will show that the lengths of γ_n in fact diverge as $n \to +\infty$ (so, in particular, they do not converge to $\ell(\gamma)$): Since $\|\dot{\gamma}_n(x)\| = \sqrt{1 + n\cos^2(nx)}$, we have that

$$\|\dot{\gamma}_n(x)\| \geqslant \sqrt{n} |\cos(nx)|.$$

Upon integration, we have

$$\begin{aligned}\ell(\gamma_n) &= \int_0^{2\pi} \|\dot{\gamma}_n(x)\| \, dx \\ &\geqslant \sqrt{n} \int_0^{2\pi} |\cos(nx)| \, dx \\ &\geqslant \sqrt{n} \sum_{k=0}^{n-1} \int_{\frac{2k\pi}{n}}^{\frac{(2k+\frac{1}{2})\pi}{n}} |\cos(nx)| \, dx\end{aligned}$$

$$\geqslant \sqrt{n}\sum_{k=0}^{n-1}\int_{\frac{2k\pi}{n}}^{\frac{(2k+\frac{1}{2})\pi}{n}} \cos(nx)\,dx$$
$$= \sqrt{n}\sum_{k=0}^{n-1}\Big[\frac{1}{n}\sin(nx)\Big]_{\frac{2k\pi}{n}}^{\frac{(2k+\frac{1}{2})\pi}{n}}$$
$$= \sqrt{n}\frac{n-1}{n} \geqslant \frac{1}{2}\sqrt{n}$$

so $\ell(\gamma_n) \xrightarrow{n\to\infty} +\infty$.

3.9 The goal of this exercise is to define the notion of intrinsic distance in a domain of $\mathbb{R}^n$ (by definition, a domain of $\mathbb{R}^n$ is an open, connected subset).

Let $\mathcal{U} \subset \mathbb{R}^n$ and $p, q \in \mathcal{U}$. Let $\mathcal{C}_{pq}$ denote the set of curves $\gamma : [a, b] \to \mathcal{U}$ that are continuous, piecewise C^1, and connect p to q. The intrinsic distance in $\mathcal{U}$ from p to q is then defined by

$$\delta_{\mathcal{U}}(p, q) = \inf\{\ell(\gamma) \mid \gamma \in \mathcal{C}_{pq}\}.$$

(a) Prove that $\mathcal{C}_{pq} \neq \varnothing$ for all $p, q \in \mathcal{U}$.

(b) Prove that $\delta_{\mathcal{U}}(p, q) \geqslant \|q - p\|$ for all $p, q \in \mathcal{U}$.

(c) Prove that $(\mathcal{U}, \delta_{\mathcal{U}})$ is a metric space.

(d) Under what condition on the domain $\mathcal{U}$ do we have $\delta_{\mathcal{U}}(p, q) = \|q - p\|$ for all $p, q \in \mathcal{U}$? (give a sufficient condition).

(e) Consider the case $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 \mid x < -1 \text{ or } y \neq 0\}$. What is the intrinsic distance between $p = (0, 1)$ and $q = (0, -1)$? Does there exist a curve of minimal length connecting p to q?

Solution. (a) Since $\mathcal{U} \subset \mathbb{R}^n$ is open and connected, any two of its points can be connected by a curve consisting of a finite number of line segments. Thus, $C_{pq} \neq \emptyset$.

(b) As we have seen in class, any piecewise C^1 curve γ starting from p and ending at q satisfies

$$\ell(\gamma) \geqslant \|p - q\|$$

(the proof we saw was for C^1 curves, but the same proof applies to the piecewise C^1 case). Therefore, considering the infimum over such curves, we get

$$\delta_{\mathcal{U}}(p, q) = \inf_{\gamma\in\mathcal{C}_{p,q}} \ell(\gamma) \geqslant \|q - p\|.$$

(c) We can readily see that $\delta_{\mathcal{U}}$ satisfies all conditions for being a distance function:

- It is symmetric, since, for any curve $\gamma : [0, 1] \to \mathcal{U}$ starting at p and ending at q (i.e. belonging to $\mathcal{C}_{p,q}$), the reversed curve $\tilde{\gamma}(t) = \gamma(1 - t)$ starts from q and ends at p (i.e. belongs to $\mathcal{C}_{q,p}$), and the two curves have the same length. Therefore, $\inf_{\gamma\in\mathcal{C}_{p,q}} \ell(\gamma) = \inf_{\gamma\in\mathcal{C}_{q,p}} \ell(\gamma)$.

- It satisfies the triangle inequality: Let $p, z, q \in \mathcal{U}$. If $\gamma_1 : [0,1] \to \mathcal{U}$ is a piecewise C^1 curve in $\mathcal{C}_{p,z}$ and $\gamma_2 : [0,1] \to \mathcal{U}$ is a piecewise C^1 curve in $\mathcal{C}_{z,q}$, then the curve

$$\gamma_1 \cup \gamma_2(t) = \begin{cases} \gamma_1(t), & 0 \leqslant t \leqslant 1, \\ \gamma(t-1), & 1 \leqslant t \leqslant 2, \end{cases}$$

  is a piecewise C^1 curve in $\mathcal{C}_{p,q}$ and satisfies $\ell(\gamma_1) + \ell(\gamma_2) = \ell(\gamma_1 \cup \gamma_2)$. Taking the infimum over curves $\gamma_1 \in \mathcal{C}_{p,z}$ and $\gamma_2 \in \mathcal{C}_{z,q}$, we obtain

$$\inf_{\gamma_1 \in \mathcal{C}_{p,z}} \ell(\gamma_1) + \inf_{\gamma_2 \in \mathcal{C}_{z,q}} \ell(\gamma_2) \geqslant \inf_{\substack{\gamma_1 \in \mathcal{C}_{p,z}, \\ \gamma_2 \in \mathcal{C}_{z,q}}} \ell(\gamma_1 \cup \gamma_2) \geqslant \inf_{\gamma \in \mathcal{C}_{p,q}} \ell(\gamma)$$

  so

$$\delta_{\mathcal{U}}(p,z) + \delta_{\mathcal{U}}(z,q) \geqslant \delta_{\mathcal{U}}(p,q).$$

- It is positive definite: For any two points p, q and any curve $\gamma \in \mathcal{C}_{p,q}$, we have $\ell(\gamma) \geqslant 0$ so, after taking the infimum over all those curves, we have $\delta_{\mathcal{U}}(p,q) \geqslant 0$. If $\delta_{\mathcal{U}}(p,q) = 0$, then there exists a sequence $\gamma_n \in \mathcal{C}_{p,q}$ such that $\ell(\gamma_n) \xrightarrow{n \to \infty} 0$. But, since $\ell(\gamma_n) \geqslant \|p - q\|$, this implies that $\|p - q\| = 0$ or, equivalently, that $p = q$.

Thus, $(\mathcal{U}, \delta_{\mathcal{U}})$ is a metric space.

(d) If $\mathcal{U}$ is convex, then for any two points $p, q \in \mathcal{U}$, the straight line segment $\gamma_s : [0,1] \to \mathbb{R}^n$, $\gamma(t) = tq + (1-t)p$ (which connects p to q) lies entirely inside $\mathcal{U}$ and, therefore, is in $\mathcal{C}_{p,q}$. Therefore.

$$\delta_{\mathcal{U}}(p,q) = \inf_{\gamma \in \mathcal{C}_{p,q}} \ell(\gamma) \leqslant \ell(\gamma_s) = \|p - q\|.$$

On the other hand, we showed above that $\delta_{\mathcal{U}}(p,q) \geqslant \|p - q\|$. Therefore, $\delta_{\mathcal{U}}(p,q) = \|q - p\|$.

(e) For $\mathcal{U}$ the given domain, any curve from $(0,1)$ to $(0,-1)$ must pass through some $(u,0)$ with $u < -1$ (any curve connecting $(0,1)$ to $(0,-1)$ must pass through $\{y = 0\}$, but if the curve is entirely inside $\mathcal{U}$, the intersection point must be at $x < -1$ because of the shape of the domain). Therefore, any such curve γ passing through $(u,0)$ must satisfy

$$\ell(\gamma) \geqslant \|(0,1) - (u,0)\| + \|(u,0) - (0,-1)\| = 2\sqrt{1 + u^2}.$$

Therefore, taking the infimum over such curves:

$$\delta_{\mathcal{U}} U((0,1),(0,-1)) \geqslant \inf_{u < -1} \left(2\sqrt{1 + u^2} \right) = 2\sqrt{2}.$$

No minimizing curve exists, as any curve passing through $(u,0)$ with $u < -1$ will have length $\geqslant 2\sqrt{1+u^2} > 2\sqrt{2}$ (any minimizing sequence of curves, however, would approach the curve in $\mathbb{R}^2$ consisting of two straight line segments, one connecting $(0,1)$ to $(-1,0)$ and one connecting $(-1,0)$ to $(0,-1)$).